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ON THE SECOND MOMENT OF THE NUMBER OF CROSSINGS BY A STATIONARY GAUSSIAN PROCESS

By Marie F. Kratz and José R. León[1]

MAP5 UMR 8145 and SAMOS-MATISSE UMR 8595, and UCV Caracas

Cramér and Leadbetter introduced in 1967 the sufficient condition

$$\frac{r''(s)-r''(0)}{s}\in L^1([0,\delta],dx),\qquad \delta>0,$$

to have a finite variance of the number of zeros of a centered stationary Gaussian process with twice differentiable covariance function r. This condition is known as the Geman condition, since Geman proved in 1972 that it was also a necessary condition. Up to now no such criterion was known for counts of crossings of a level other than the mean. This paper shows that the Geman condition is still sufficient and necessary to have a finite variance of the number of any fixed level crossings. For the generalization to the number of a curve crossings, a condition on the curve has to be added to the Geman condition.

1. Introduction and main result. Let $X=\{X_t,\ t\in\mathbb{R}\}$ be a centered stationary Gaussian process. Its correlation function r is supposed to be twice differentiable and to satisfy on $[0,\delta]$, with $\delta>0$,

$$r(\tau)=1+\frac{r''(0)}{2}\tau^2+\theta(\tau) \tag{1}$$
$$\text{with } \theta(\tau)>0, \frac{\theta(\tau)}{\tau^2}\to 0, \frac{\theta'(\tau)}{\tau}\to 0, \theta''(\tau)\to 0,\ \text{as } \tau\to 0.$$

The nonnegative function L defined by $\theta''(\tau):=\tau L(\tau)$ will be referred to as the Geman function.

Let us consider a continuous differentiable real function ψ and let us define, as in [2], the number of crossings of the function ψ by the process

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X on an interval $[0,t]$ ($t \in \mathbb{R}$), as the random variable $N_t^\psi = N_t(\psi) = \#\{s \leq t : X_s = \psi_s\}$.

The number N_t^ψ of ψ-crossings by X can also be seen as the number of zero crossings $N_t^Y(0)$ by the nonstationary (but stationary in the sense of the covariance) Gaussian process $Y = \{Y_s,\ s \in \mathbb{R}\}$, with $Y_s := X_s - \psi_s$, that is, $N_t^\psi = N_t^Y(0)$.

Regarding the moments of the number of crossings by X, one of the most well-known first results was obtained by Rice [9] for a given level x, namely

$$\mathbb{E}[N_t(x)] = \ te^{-x^2/2} \ \sqrt{-r''(0)} \ /\pi.$$

This equality was proved two decades later by Itô [7] and Ylvisaker [11], providing a necessary and sufficient condition to have a finite mean number of crossings:

$$\mathbb{E}[N_t(x)] < \infty \quad \Longleftrightarrow \quad -r''(0) < \infty.$$

Also in the 1960s, following on the work of Cramér, generalization to curve crossings and higher-order moments for $N_t(\cdot)$ were considered in a series of papers by Cramér and Leadbetter [2] and Ylvisaker [12].

Moreover, Cramér and Leadbetter [2] provided an explicit formula for the second factorial moment of the number of zeros of the process X, and proposed a sufficient condition on the correlation function of X in order to have the random variable $N_t(0)$ belonging to $L^2(\Omega)$, namely

$$\text{If } L(t) := \frac{r''(t) - r''(0)}{t} \ \in L^1([0,\delta], dx) \qquad \text{then } \mathbb{E}[N_t^2(0)] < \infty.$$

Geman [6] proved that this condition was not only sufficient but also necessary:

$$\mathbb{E}[N_t^2(0)] < \infty \quad \Longleftrightarrow \quad L(t) \ \in L^1([0,\delta], dx) \qquad \text{(Geman condition)}. \tag{2}$$

This condition held only when choosing the level as the mean of the process.

Generalizing this result to any given level x and to some differentiable curve ψ has been subject to some investigation and nice papers, such as the ones of Cuzick [4, 5] proposing sufficient conditions. But to get necessary conditions remained an open problem for many years. The solution of this problem is enunciated in the following theorem.

Theorem.

(1) *For any given level x, we have*

$$\mathbb{E}[N_t^2(x)] < \infty \quad \Longleftrightarrow \quad \exists\, \delta > 0, L(t) = \frac{r''(t) - r''(0)}{t} \in L^1([0,\delta], dx)$$

(*Geman condition*).

(2) *Suppose that the continuous differentiable real function* ψ *is such that*

$$\exists\,\delta>0 \qquad \int_0^\delta \frac{\gamma(s)}{s}\,ds<\infty \tag{3}$$

where $\gamma(\cdot)$ *is the modulus of continuity of* $\dot{\psi}$.

Then

$$\mathbb{E}[N_t^2(\psi)]<\infty \quad\Longleftrightarrow\quad L(t)\in L^1([0,\delta],dx).$$

REMARK. This smooth condition on ψ is satisfied by a large class of functions which includes in particular functions whose derivatives are Hölder.

Finally let us mention the work of Belyaev [1] and Cuzick [3, 4, 5] who proposed some sufficient conditions to have the finiteness of the kth (factorial) moments for the number of crossings for $k\geq 2$. When $k\geq 3$, the difficult problem of finding necessary conditions when considering levels other than the mean is still open.

2. Proof. Generalizing the formula of Cramér and Leadbetter ([2], page 209) concerning the zero crossings, the second factorial moment M_2^ψ of the number of ψ-crossings can be expressed as

$$\begin{aligned} M_2^\psi = {}& \int_0^t\int_0^t\int_{R^2}|\dot{x}_1-\dot{\psi}_{t_1}||\dot{x}_2-\dot{\psi}_{t_2}| \\ & \times p_{t_1,t_2}(\psi_{t_1},\dot{x}_1,\psi_{t_2},\dot{x}_2)\,d\dot{x}_1\,d\dot{x}_2\,dt_1\,dt_2, \end{aligned} \tag{4}$$

where $p_{t_1,t_2}(x_1,\dot{x}_1,x_2,\dot{x}_2)$ is the density of the vector $(X_{t_1},\dot{X}_{t_1},X_{t_2},\dot{X}_{t_2})$ that is supposed nonsingular for all $t_1\neq t_2$. The formula holds whether M_2^ψ is finite or not.

We also have

$$\begin{aligned} M_2^\psi = {}& 2\int_0^t\int_{t_1}^t p_{t_1,t_2}(\psi_{t_1},\psi_{t_2}) \\ & \times \mathbb{E}[|\dot{X}_{t_1}-\dot{\psi}_{t_1}||\dot{X}_{t_2}-\dot{\psi}_{t_2}||X_{t_1}=\psi_{t_1},X_{t_2}=\psi_{t_2}]\,dt_2\,dt_1, \end{aligned} \tag{5}$$

where $p_{t_1,t_2}(x_1,x_2)$ is the density of (X_{t_1},X_{t_2}).

From now on, let us put $t_2=t_1+\tau$, $\tau>0$.

The method used to prove that the Geman condition keeps being the sufficient and necessary condition to have M_2^ψ finite can be sketched into three steps.

The first one consists in using the following regression model to compute the expectation in M_2^ψ:

$$(R)\qquad \begin{aligned}\dot X_{t_1} &= \zeta + \alpha_1(\tau)X_{t_1} + \alpha_2(\tau)X_{t_1+\tau},\\ \dot X_{t_1+\tau} &= \zeta^* - \beta_1(\tau)X_{t_1} - \beta_2(\tau)X_{t_1+\tau},\end{aligned}$$

where (ζ,ζ^*) is jointly Gaussian such that

$$\mathrm{Var}(\zeta) = \mathrm{Var}(\zeta^*) := \sigma^2(\tau) = -r''(0) - \frac{r'^2(\tau)}{1-r^2(\tau)}, \tag{6}$$

$$\rho(\tau) := \frac{\mathrm{Cov}(\zeta,\zeta^*)}{\sigma^2(\tau)} = \frac{-r''(\tau)(1-r^2(\tau)) - r'^2(\tau)r(\tau)}{-r''(0)(1-r^2(\tau)) - r'^2(\tau)}, \tag{7}$$

and where

$$\alpha_1 = \alpha_1(\tau) = \frac{r'(\tau)r(\tau)}{1-r^2(\tau)}; \qquad \alpha_2 = \alpha_2(\tau) = -\frac{r'(\tau)}{1-r^2(\tau)}$$

$$\beta_1 = \beta_1(\tau) = \alpha_2(\tau); \qquad \beta_2 = \beta_2(\tau) = \alpha_1(\tau).$$

In the second step, the expectation, formulated in terms of ζ and ζ^*, will be expand into Hermite polynomials. Recall that the Hermite polynomials $(H_n)_{n\geq 0}$, defined by $H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$, constitute a complete orthogonal system in the Hilbert space $L^2(\mathbb{R},\varphi(u)\,du)$, φ denoting the standard normal density.

Finally, this Hermite expansion will allow us to find, in an easier way, lower and upper bounds for M_2^ψ. Nevertheless, it will required a fine study in the neighborhood of 0, on one hand on the correlation function r of X and its derivatives, showing in particular the close relation between the existence of the Geman function L and the existence of $r^{(iv)}(0)$, on the other hand, on the correlation function ρ of the r.v. ζ and ζ^* of the model (R). It will be presented in the two first lemmas below. Moreover, since the bounds will be expressed in terms of the variance $\sigma^2(\tau)$ of the r.v. ζ (or ζ^*), an interesting lemma (see Lemma 3 below) will show that the behavior of L is closely related to the behavior of $\sigma^2(\tau)$.

Lemma 1.

(i) *If* $r^{(iv)}(0) = +\infty$, *then* $\lim_{\tau\to 0}\frac{L(\tau)}{\tau} = +\infty$.
(ii) *If* $r^{(iv)}(0) < +\infty$, *then* $\lim_{\tau\to 0}\frac{L(\tau)}{\tau} = \frac{r^{(iv)}(0)}{2}$.

Lemma 2. *For* τ *belonging to a neighborhood of* 0*:*

(i) $|\frac{r'(\tau)}{\sigma(\tau)}|$ *is bounded;*
(ii) $\rho(\tau) \leq 0$.

LEMMA 3. *For τ belonging to a neighborhood of* 0*:*

(i) $\frac{\sigma^2(\tau)}{\tau} \le L(\tau) \le (2+C)\frac{\sigma^2(\tau)}{\tau}$, *with* $C \ge 0$*;*
(ii) *For* $\delta > 0$, $\int_0^\delta \frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}\,d\tau < \infty \Leftrightarrow \int_0^\delta L(\tau)\,d\tau < \infty$ *(Geman condition).*

The proofs of the lemmas are given in [8].

To illustrate the method, we will present the complete proof when considering a fixed level x. For the case of curve-crossings, you can refer to [8].

So suppose $\dot{\psi}_s = 0$ and $\psi_s \equiv x, \forall\, s$.

Let C be a positive constant which may vary from equation to equation. By using the regression (R), M_2^x can be written as

$$M_2^x = 2\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)A(m,\rho,\tau)\,d\tau,$$

where

$$A(m,\rho,\tau) := \mathbb{E}\left|\left(\frac{\zeta}{\sigma(\tau)} + \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\left(\frac{\zeta^*}{\sigma(\tau)} - \frac{r'(\tau)}{(1+r(\tau))\sigma(\tau)}x\right)\right|,$$

and $p_\tau(x,x) := p_{0,\tau}(x,x)$.

Note that

$$M_2^x \ge M_2^{x,\delta} := 2\int_0^\delta (t-\tau)p_\tau(x,x)\sigma^2(\tau)A(m,\rho,\tau)\,d\tau, \qquad \delta \in [0,\tau]. \tag{8}$$

Now, by using Mehler's formula (see, e.g., [10]), we have

$$A(m,\rho,\tau) = \sum_{k=0}^{\infty} a_k(m)a_k(-m)k!\rho^k(\tau) \qquad \text{where } m = m(\tau) := \frac{r'(\tau)x}{(1+r(\tau))\sigma(\tau)},$$

$|m| = |m(\tau)|$ being bounded because of (i) of Lemma 2, and $a_k(m)$ are the Hermite coefficients of the function $|\cdot - m|$, given by

$$\begin{aligned} a_0(m) &= \mathbb{E}|Z-m| \qquad Z \text{ being a standard Gaussian r.v.} \\ &= m[2\Phi(m)-1] + \sqrt{\frac{2}{\pi}}e^{-m^2/2}, \\ a_1(m) &= (1-2\Phi(m)) = -\sqrt{\frac{2}{\pi}}\int_0^m e^{-u^2/2}\,du \end{aligned}$$

and

$$a_l(m) = \sqrt{\frac{2}{\pi}}\frac{1}{l!}H_{l-2}(m)e^{-m^2/2}, \qquad l \ge 2.$$

Let us show that $M_2^x < \infty$ under the Geman condition.

Since by Cauchy–Schwarz inequality

$$|A(m,\rho,\tau)| \le \sum_{k=0}^{\infty} |a_k(m)a_k(-m)|k! \le (\mathbb{E}[(Y-m)^2]\mathbb{E}[(Y+m)^2])^{1/2},$$

with Y a standard normal r.v., there follows

$$M_2^x \le I_2 := 2\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)(a_0(m)a_0(-m)+1+m^2)\,d\tau.$$

Hence, m^2 being bounded, we obtain $I_2 \le C\int_0^t (t-\tau)p_\tau(x,x)\sigma^2(\tau)\,d\tau$.

The study of this last integral reduces to the one on $[0,\delta]$ because of the uniform continuity outside of a neighborhood of 0, so we can conclude that it is finite if $L \in L^1[0,\delta]$, by using Lemma 3(ii).

Let us look now at the reverse implication.

Suppose that $M_2^x < \infty$, and so, via (8), that $M_2^{x,\delta} < \infty$.

Let us compute $A(m,\rho,\tau)$ and bound it below.

By using the parity of the Hermite polynomials and the sign of ρ given in (ii) of Lemma 2, we obtain

$$\begin{aligned} A(m,\rho,\tau) &= a_0^2(m) + |\rho(\tau)|a_1^2(m) + \sum_{k=1}^{\infty} a_{2k}^2(m)(2k)!\rho^{2k}(\tau) \\ &\quad + |\rho|\sum_{k=1}^{\infty} a_{2k+1}^2(m)(2k+1)!\rho^{2k}(\tau) \\ &\ge a_0^2(m) = \left(-ma_1(m) + \sqrt{\frac{2}{\pi}}e^{-m^2/2}\right)^2 \\ &\ge \frac{2}{\pi}e^{-m^2} \ge C \qquad (\text{since } |m|<\infty). \end{aligned}$$

Hence

$$M_2^{x,\delta} \ge C\int_0^\delta (t-\tau)p_\tau(x,x)\sigma^2(\tau)\,d\tau \ge C\int_0^\delta (t-\tau)\frac{\sigma^2(\tau)}{\sqrt{1-r^2(\tau)}}\,d\tau.$$

An application of Lemma 3(ii), yields that $M_2^{x,\delta} < \infty$ implies the Geman condition.

The proof of the general case follows the same approach. It requires also to use Taylor formula for ψ and to introduce the modulus of continuity of $\dot{\psi}$ to express the expectation in the integrand of M_2^ψ into two terms, one on which will be applied the described method, the other related to the modulus of continuity of $\dot{\psi}$, which is bounded thanks to the condition (3) of the theorem (for more details, see [8]).

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U.F.R. de Mathématiques et Informatique
Université René Descartes, Paris V
45 rue des Saints-Pères
75270 Paris Cedex 06
France
E-mail: kratz@math-info.univ-paris5.fr
URL: www.math-info.univ-paris5.fr/~kratz

Escuela de Matemática
Facultad de Ciencias
Universidad Central de Venezuela
A.P. 47197 Los Chaguaramos
Caracas 1041-A
Venezuela
E-mail: jleon@euler.ciens.ucv.ve
URL: euler.ciens.ucv.ve/